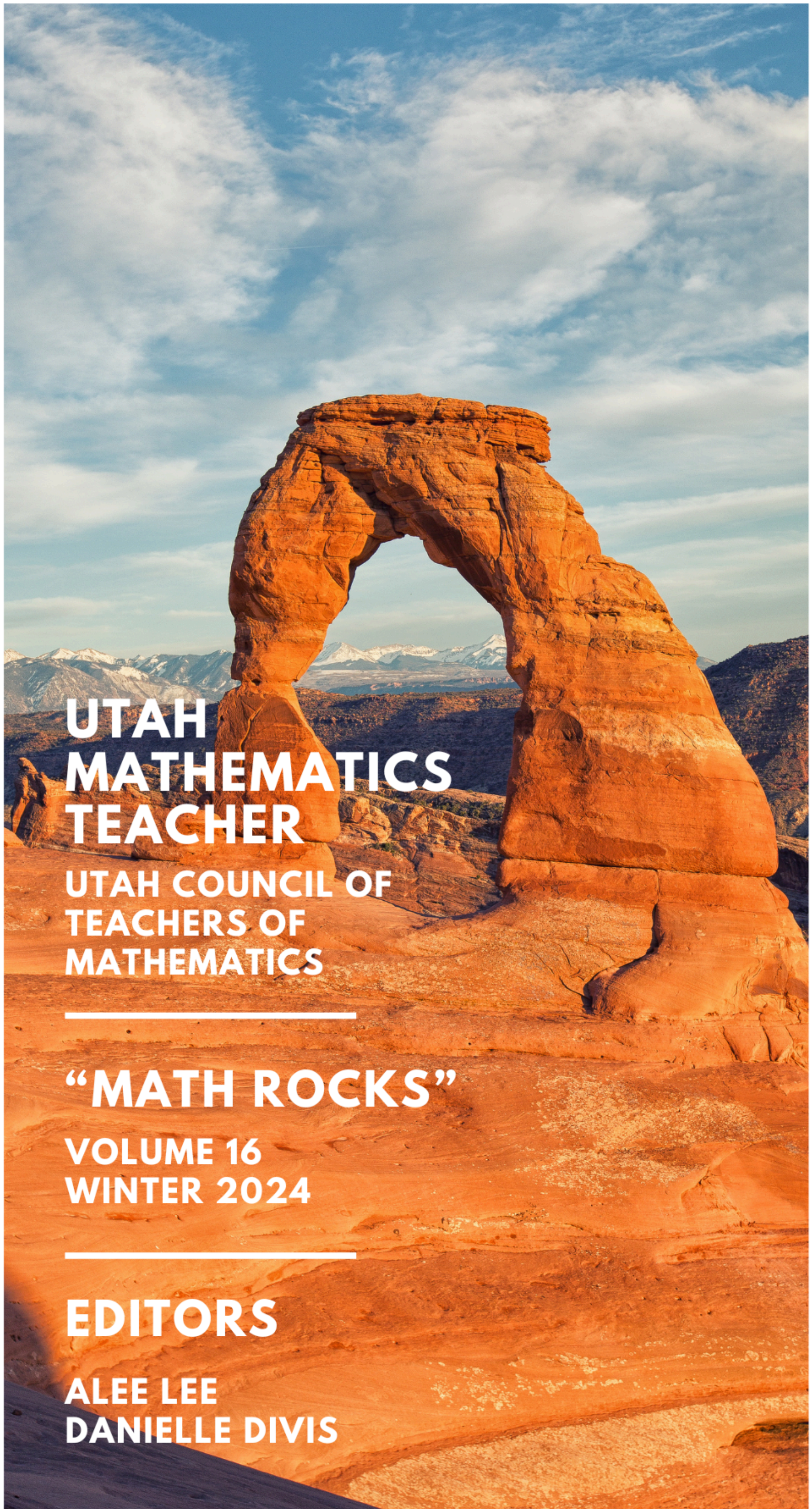
MATH ROCKS



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"MATH ROCKS"

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CALL FOR ARTICLES

The Utah Mathematics Teacher seeks articles on issues of interest to mathematics educators, especially K-12 classroom teachers in Utah. All are encouraged to contribute articles and opinions for any section of the journal. Manuscripts, including tables and figures, should be typed in Microsoft Word and submitted electronically as an e-mail attachment to Alees Lee at aleeslee@weber.edu or Danielle Divis at danielledivis21@gmail.com. A cover letter containing author's name, address, affiliations, phone, e-mail address and the article's intended audience should be included.

UTAH MATHEMATICS TEACHER

Volume 16, Winter 2024

"Math Rocks"

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UCTM President's Message

Rebecca Roche, Granger High School

One of my favorite things to do is travel. Especially during the winter months after Christmas, it's fun to dream of new places to visit and explore. I love planning where we go, what we see, and all the yummy food stops. I want to finish the trip feeling like I fully immersed myself and did not hold back. Travel is also much more fun with travel buddies. If your friends share your vision for the trip, bring positivity and life to the adventures, and are there to support you if anything crazy happens on your trip, then they are keepers. No matter how far from the original itinerary you stray, it's possible to pivot and redeem unexpected situations.

Teaching is so similar to traveling. There are amazing picture worthy moments, seemingly millions of factors that you can and cannot control, teams that surround us and can make life way better or wayyy worse, and so many moving parts it can get hard to keep track of it all. Can math class be a fun destination? We can certainly plan an itinerary that excites and informs our visitors so they want to keep coming back for more! As a high school math teacher and daughter of an elementary teacher, I know the work that goes into preparing lessons for all age levels. Thank you for the intentionality and commitment you have given to this profession and the time and effort you put into your work.

We joined math education because, on some level, we not only believe that "math rocks" but that it's important for our students to recognize that, as well. Although each of our journeys in education are unique, when we surround ourselves with people who are passionate, positive, and persistent in their pursuit of excellence, it is possible to create spaces that inspire growth in us and our students. Thank you for embracing the chaos and creating meaningful learning experiences, even when things go sideways and the unexpected pops up. Thank you for the way you elevate your surroundings. And thank you for being the type of person that is willing to go on this journey with us.

Letter from the Editors

Alees Lee, Ph.D.
Danielle Divis, Ph.D.

As 2024 begins, the editors of Utah Mathematics Teacher were pleased with the response and support for continuing a winter publication of the journal! This year, we received many wonderful submissions including articles that highlight mathematical models and strategies for engaging all learners.

The UCTM 2024 conference theme is Math Rocks. Being a recent transplant to the beautiful state of Utah, I took the theme as a sign to visit the infamous Arch this summer and wasn't disappointed – nature truly rocks. As I reflect back on the breathtaking views in Moab, I am reminded that the creation of beautiful things takes time, persistence, and all kinds of weather. I couldn't help but see the parallels between the work of teaching and the formation of things in nature. Most of us became mathematics teachers because of our love for math. We think math rocks (and it definitely does!), but in doing the work of teaching the love of math can often become hidden and secondary to all the other elements of our jobs. We can lose sight of what pulled us into the profession to begin with. I want to remind us that math does, in fact, rock even as we experience all kinds of weather.

Within this journal, you will find articles that encourage you to see mathematical models in new ways, offer you strategies for engaging with your students, and to reflect on how to think more deeply about elements of your job that go beyond the act of teaching. As you read through the line-up of articles, I want to you to feel inspired to make math rock again.

We hope you enjoy this unique selection of articles and that you use them to find continued joy in mathematics and mathematics teaching. In addition, please consider submitting your own articles or serving as a reviewer for further publications.

Note. Any mistakes are the sole responsibility of the editor and will be remedied in the online journal. Please send corrections to aleeslee@weber.edu or danielledivis21@gmail.com.

Teaching Culturally Diverse Students in a Mathematics Classroom: Ideas Composed from a Summary of the Current Literature

Christine Walker, Utah Valley University

Abstract

What are the best strategies for teaching mathematics to English Language Learners (ELLs) and immigrant students? This literature review presents an overview of the current literature focused on key strategies that support English Language Learners in connecting mathematics to their world. Having ELL-specific math strategies to implement in math classrooms can help students develop math proficiency while building their language proficiency, in turn helping them feel more confident and comfortable in their mathematics classrooms.

Keywords: Mathematics Methods Courses; Preservice teachers; Teacher education; Inservice Math Teachers; English Language Learners

As a University Professor of Mathematics, focused on preparing preservice secondary math teachers, questions have arisen in preservice courses, and in student teaching/intern observations regarding the teaching of mathematics to those who have a language barrier. Preservice teachers expressed a strong desire to provide English Language Learners with the foundation needed to demonstrate mathematical thinking. Thus, the purpose of this literature review is to present an overview of the current literature focused on the best strategies for teachers to teach mathematics to English Language Learners.

English Language Learners (ELLs) are students who are not yet able to communicate fluently in English, which is sometimes referenced as a linguistic barrier. Barriers for ELLs have been categorized as linguistic and cultural, which surface during instruction and are present in the curriculum (Encore!!!, n.d.). The challenges to language barriers can lead to potential misunderstandings, affect personal relationships, impact student work, and can result in a failure to keep up with coursework.

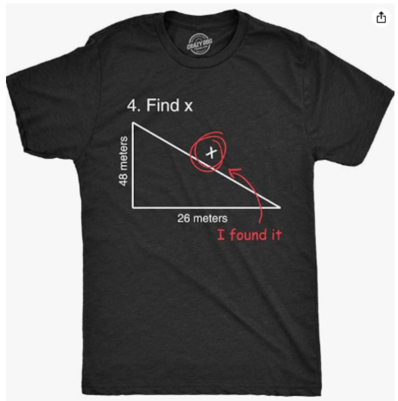
In 2022, Utah had a total enrollment of 674,650 in the public school system. Of those 674,650 enrolled students, 59,176 were categorized as English Language Learners. This constituted 8.8% of the population which was a 6.5% increase over the previous years' data and points to a growth pattern in the public-school population of ELLs (Peterson, 2022). This results in math teachers, who are not trained ELL teachers, with the sometimes overwhelming task of providing effective and engaging mathematics instruction to students who they might not even know are ELLs. A review of the research resulted in key strategies for math teachers to help ELLs succeed in learning mathematics.

Strategies

Teaching strategies (instructional strategies) are often thought of as the methods that teachers use to deliver content to keep students engaged. It ranges from techniques, procedures, and processes utilized during instruction time.   In college and university departments of education, preservice teachers learn the theory and practice of education in addition to the foundation of mathematics pedagogy in their content methods courses. In recent years, colleges of education have added coursework specific to multicultural instruction and English as a Second Language (ESL) due to the increase in ELLs in public schools. For example, the National Center for Education Statistics (NCEC) determined that the percentage of public school students in the United States who were ELLs in the fall of 2020 was 5.0 million students or a 10.3% increase from the fall of 2010 (U.S. Department of Education, 2023). However, many teachers who did not have the opportunity to take coursework specific to multicultural instruction and English as a Second Language (ESL) often feel "…largely untrained to work with ELL students…" (Durgunoglu & Hughes, 2010).

Due to the significant increase in the number of ELL students, both in Utah and across the U.S., increasing attention and research have focused on providing teachers with tools and strategies to help ELLs succeed in mainstream classrooms. As specified in the research, when it comes to learning mathematics, students who have a language barrier, face a more challenging problem of learning the language and the language of mathematics simultaneously.

As an example of the challenges ELLs face learning the language and the language of mathematics simultaneously, Kristina Robertson writes about a famous math problem that is often imprinted on “gag” math t-shirts for sale. It involves a problem where a student is to “find x.”



(Fashion, 2023)

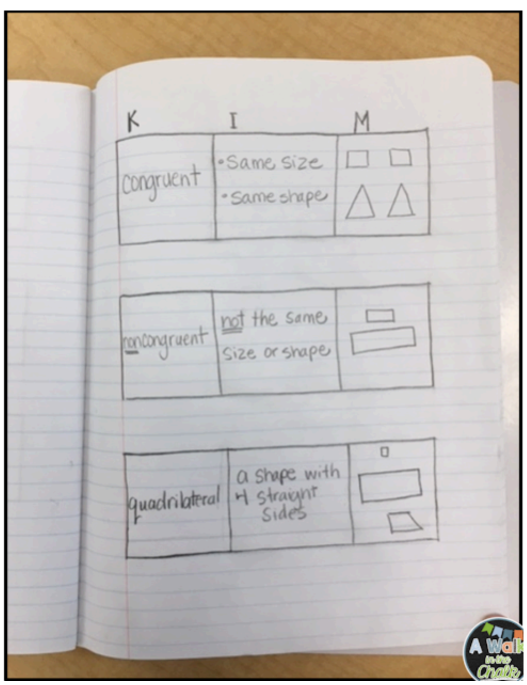
She writes, “The student obviously knew the meaning of the word "find" because he/she "found" it on the page and circled it. The student even put a note on the page to help the teacher in locating the lost "x". The student understood the meaning of "find" in one context, but not in the appropriate mathematical context” (Robertson, n.d.).

As noted in Robertson’s example, students, in particular ELL students, often lack familiarity with mathematics vocabulary and context.   The good news is that there are key strategies to help ELLs overcome these obstacles and have been shown by research and experience to have a positive impact on the mathematics achievement of ELL students. The current literature indicates key strategies focused on the areas of mathematics vocabulary, changes to instructional routines, and teacher talk/partner talk.

Mathematics Vocabulary

Students often come with some basic mathematical vocabulary such as add, subtract, and counting numbers for example one, two, three, etc. Focusing on the words and phrases, and in best cases accompanying the vocabulary words with illustrations, can help students build understanding by making connections between the words and the mathematics. ELLs also need opportunities to use the new vocabulary in a variety of ways.

Vocabulary Banks. Vocabulary banks are one strategy for implementing new vocabulary in the classroom. It is especially important when word problems are posed to help ELLs understand the context. Vocabulary banks can be employed in worksheets, posting a chart of foundational mathematics vocabulary words and phrases in the room (Willig, Bresser, Melanese, Sphar, & Felux, 2014), or “playing vocabulary games, reading math readers” (Blankman, 2021). Another version of a vocabulary bank is the strategy called the K.I.M Strategy, which stands for Keyword, Important information, and Memory clue. It has been defined as a “low prep, high yield” vocabulary-building strategy and involves creating a three-column chart, composed of the keyword, a definition of the word or information that matches the keyword, and a memory clue such as a picture or visual representation. As an example:



(Vibas, 2023)

Sentence Frames. Creating and posting lesson-specific or vocabulary-specific sentence frames can check students' comprehension of mathematical vocabulary and help ELLs begin to form complete mathematical sentences (Robertson, n.d.; Wilburne, Marinak, & Strickland, 2011). It allows ELLs to practice speaking in a low-risk setting and supports the introduction of new vocabulary in a context to practice the vocabulary. Sentence frames can be categorized in a variety of ways, such as describing, comparing, categorizing, sequencing, predicting, and drawing conclusions to name just a few (Blankman, 2021). An example would be "The answer is __________ degrees because it is a __________ triangle".

Reinforce Vocabulary through Verbal and Nonverbal Responses. Nonverbal responses such as thumbs up or down can be used to check understanding (Willig, Bresser, Melanese, Sphar, & Felux, 2014), as well as verbal responses, such as having all students verbally repeat a new word out loud. This ensures ELLs can hear the correct pronunciation of the word. Aligning the new word with a visual chart and illustration helps to connect the grammar with an image-building connection between the new word and a visual. These are often referred to as Math Word Walls (McKay, n.d.; Vibas, 2023).

On the National Council of Teachers of Mathematics (NTCM) website, under Classroom Resources is a link for 'Activities with Rigor and Coherence', referred to as ARCs. One activity called Triangle Congruence, Lesson 1 of 4, titled "Congruent Halves", has students create two congruent halves from one polygon by using transformations. In the introduction to the lesson, the author writes "For English language learners, it may be useful to print and have on a "Word Wall" the images you show of a rotation, reflection, and translation" (Ray-Riek, Duarte, & Baker, n.d.).

Instructional Routines

The publication *Principles to Action* articulates a view of instructional routines that are known to support the development of mathematical proficiency (NCTM, 2014). The instructional routines focus on eight mathematics teaching practices that "provide a framework for strengthening the teaching and learning of mathematics…which represent a core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics" (p.10). They also represent an effective typical day in a mathematics classroom, such as establishing a goal (in some districts, this is called an "I can" statement); implementing a task/posing a question; facilitating discourse/using and connecting representations and eliciting student thinking; building procedural fluency from conceptual understanding/task; and supporting struggle.

Kendall Hunt Publishing Company partnered with Illustrative Mathematics to implement specific instructional routines "known as Mathematical Language Routines (MLR)…developed by the Stanford University UL/SCALE team…(which are) supports for students with emerging English language proficiency" (Illustrative Mathematics, n.d.). The eight MLRs are instructional strategies for teachers to implement in classrooms to meet the needs of all students with a focus on three of the eight that are most effective for ELLs.

Three Reads. The "three reads" strategy involves having a student read a math scenario/problem three times, each time having a different goal in mind. The first read is to have the student understand the context of the problem and not focus on the question at the end, often having the teacher read the problem orally. This can even include a whole class discussion about what the problem might be about. The second read is designed to understand mathematics and is often completed with a partner or whole class. The second read is to identify quantities and how they relate to the problem which involves writing student ideas on a whiteboard. The third read of the problem is where the context of the problem is read one last time and the focus is on the actual question posed. A discussion usually takes place regarding how students might go about solving the problem and can occur in partner pairs or as a whole class (Herbert, n.d.; Illustrative Mathematics, n.d.).

Stronger and Clearer Each Time. The Stronger and Clearer Each Time strategy involves a scenario where students begin with a prompt or problem and write what they know about the problem. Students are encouraged to draw a diagram and use words or phrases. Once this stage is complete, students are paired with a partner to share their own thinking and get feedback. Then the roles are reversed, and they do the same for their partner. In both circumstances, students write down things they heard from their partner, incorporate the new information from their partner, and then revise what they had initially written. This can continue with 2-3 different partners, with the final draft of the problem clearer and more precise (Illustrative Mathematics, n.d.). The routine supports ELLs in using and improving their English and mathematics language in a low-risk environment before sharing with the whole class.

Information Gap Cards. Information Gap Cards is an activity that involves two students or partner pairs. The first student has a data card, and the second student has a problem card. The cards can be task cards, math puzzles, or content cards. Each card has information about the problem, but not enough information to solve the problem without working together. As an example:

Info Gap: Angle Finding

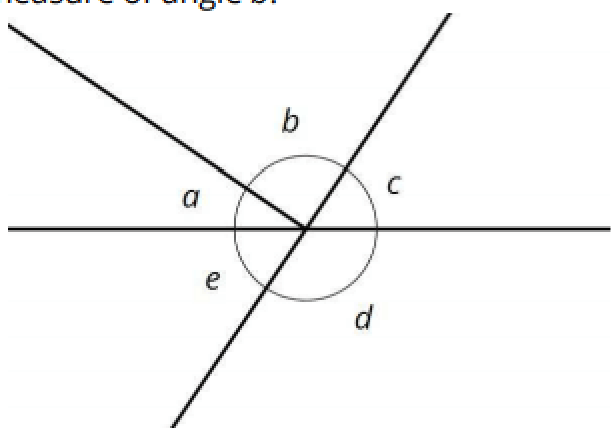
Data Card 1

- Angles c and e are vertical angles.
- Angles a and c are complementary angles.
- The measure of angle d = 124º.
- The measure of angle c = 56º.

Info Gap: Angle Finding

Problem Card 1

Find the measure of angle *b*.



(Estrella, n.d.)

As with the Stronger and Clearer Each Time strategy, the routine encourages ELLs to use mathematical language to communicate with their partner improving their English and mathematics language simultaneously. It also reinforces mathematical vocabulary, pronunciation, and meaning (Illustrative Mathematics, n.d.; Estrella, n.d.).

Manipulatives & Technology. Manipulatives are a tool that can help students progress from a concrete circumstance in mathematics to abstract reasoning in solving problems (Willig, Bresser, Melanese, Sphar, & Felux, 2014). They are often engaging and highly active, where students can create situations in a low-risk environment. Manipulatives can help students understand the context and visualize the problem to help build their vocabulary and comprehend what they are being asked to solve (Robertson, n.d.; Vibas, 2023). It is especially beneficial to help students who understand mathematics but have learned a different algorithm or model. Richard Blankman demonstrates this idea by showing what division looks like to a Venezuelan, U.S., and French student:

(Blankman, 2021)

Teacher Talk/Partner Talk

Teachers can help ELLs think and process mathematical ideas by utilizing strategies such as clear articulation, speaking slowly, reading word problems several times, and placing more emphasis on new vocabulary words or questions by reading and writing them on the board. Words are not the only method of communication. Teacher talk can be conveyed through gestures, facial expressions, and visual aids (Encore!!!, n.d.).

Designing Leveled Questions. Depending on the ELLs level of understanding, designing open-ended leveled questions can begin a conversation or confirm a student's mathematical understanding. Leveled questions could begin with a pointing question such as, "Show me the circle", progressing to a yes/no question "Is one fraction larger than the other fraction", and finally to a short-answer question such as, "Is a square a polygon?" (Willig, Bresser, Melanese, Sphar, & Felux, 2014)

Prompts. Prompts are another version of leveled questions with an open-ended aspect to solicit a more in-depth response. It can be used as a low-risk way to further the mathematical conversation. A prompt in a small group setting is most useful to help ELLs articulate an idea such as “You said a square is a polygon because…” or “You figured out one fraction was larger than the other fraction by…” (Willig, Bresser, Melanese, Sphar, & Felux, 2014).

Grouping Students/Working together. Background and context knowledge in mathematics plays a critical role in understanding how to start mathematics problems. Brenda Krick-Morales writes, “Word problems in mathematics often pose a challenge because they require that students read and comprehend the text of the problem, identify the question that needs to be answered, and finally create and solve a numerical equation. Many ELLs may have difficulty reading and understanding the written content in a word problem. If a student is learning English as a second language, he might not yet know key terminology needed to solve the equation” (Krick-Morales, n.d.; Robertson, n.d.).

In *The Utah Middle School Math Project* textbook for Grade 7, the ‘4.0 Anchor Problem: Tasting Lemonade’ (Chapter 2), asks students to determine the most “lemony” recipe (A University of Utah Partnership Project for 7th and 8th Grade Math, 2023). The question uses the word cup, which can mean two things. A cup can mean something to drink from, however, in the context of the problem a cup is used as a measurement (Picchi Cwynar & Hewett, n.d.; Robertson, n.d.). Grouping students or pairing with a partner who is composed of a mix of first and second-English language learners, vocabulary words with double meanings can be discussed (i.e. “rational”), and their meanings made clear, enabling ELLs the ability to understand and move forward in understanding and solving the problem. As an example, in a document published by the Board of Cooperative Educational Services (BOES), a list of polysemous words was created to demonstrate what words were confusing for ELLs.

Word	Meaning in Everyday Life	Meaning in Math
angle	a viewpoint or standpoint	In geometry, it's the space within two lines.
mean	(adj) offensive* (v) to intend*	An average
table	furniture	An arrangement of numbers, symbols or words to exhibit facts or relations
volume	loudness	Amount, total of
tree	a plant	Tree diagrams
area	a space or surface	The quantitative measure of a plane or curved surface
root	the underground part of a plant	The quantity raised to the power1/r
gross	offensive, disgusting	The total income from sales
operation	medical surgery	A math process, addition, multiplication…
domain	territory	The set of values assigned
degree	diploma	The sum of the exponents of the variables in a algebraic term
expression	a look indicating a feeling	A symbol representing a value
order	a command.	In algebra, the degree
power	the ability to do something, strength	the product obtained by multiplying a quantity by itself one or more times (3 diff meanings)
Odd	bizarre	leaving a remainder of 1 when divided by 2. Numbers such as 3, 5…
even	smooth, straight	a number divisible by two

(BOCES, n.d.)

Conclusions

Whether you're a teacher of ELLs or not, implementing key strategies in math classrooms can help students develop math proficiency while building language proficiency, in turn helping all students feel more confident and comfortable in their mathematics classrooms. Implementing ELL-specific math strategies can help ELLs connect math to their world. It is highly recommended that employing some of these key strategies in classroom instruction can be transformative for ELLs, as well as any mathematics student. What works for one teacher, may not work for others, and can look different for different contexts and content. And not every strategy will work for every teacher, so why not "try some on"?

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Leveraging Picture Books to Engage Elementary Students in Mathematical Problem Solving

Nicole Gearing, Andria Disney, and Andrew Goodman, Utah Valley University

Today's grade level meeting left me with a challenge. National Kindness Week was quickly approaching, and the school counselor tasked each grade level to focus as much of our instruction on social emotional learning tasks - specifically using themes in picture books by Susan Verde. Fourth grade was assigned I Am One by Susan Verde. We read the book at the meeting, and we all loved it! It was a thoughtful and inspiring story about how every movement and change starts with a single, intentional act. One purposeful act is often all it takes to start something beautiful in your school, your community, your world. Our ELA instruction could easily be focused on poetry with students studying the verses in the book and then writing their own. Our social studies instruction could focus on how one intentional act can positively impact the state's physical environment. Even PE was on board by practicing mindfulness and meditation through yoga. It was my responsibility to fit these National Kindness Week themes into our math instruction somehow. One is a number, of course…but beyond that, I had no clue.

Like the teacher in this vignette, using picture books in language arts and social studies comes naturally to many elementary teachers, but thinking about how to connect a picture book to mathematics may be more challenging. This article explores the reasons why using picture books in mathematics instruction is helpful for student learning. It also offers concrete steps for how to implement this instructional practice into your teaching, and shares some ideas and resources to support you in leveraging picture books to engage students in mathematical problem solving.

Background and Research

Literature, especially picture books, creates meaningful contexts to engage students in mathematical problem solving (Shatzer, 2008). The images in picture books help students make visual connections to mathematics concepts. They can also pique students' curiosity, leading them to articulate mathematical notices and wonders that can then be explored and/or solved. Additionally, stories create a familiar context that helps students access mathematics and see math within their own lives.

When we think about using picture books as a basis for solving math problems, we often think we must use books that feature explicit, math-related content. But, consider how you might use any book as inspiration for problem solving. Hintz and Smith (2013) offer a helpful framework for classifying the types of books you can use as the basis for a math lesson. First, there are *text-dependent* books, where you must understand a math concept in order to understand the story, like *Two of Everything*, *The Lion's Share*, or *The Greedy Triangle,* which require knowledge of doubling, fractions, and geometric shapes, respectively, to understand the story. Next, there are *idea-enhancing* books, where math is not the central focus of the story but they offer natural places to deepen student understanding of a math concept. Examples of idea-enhancing books include *Extra Yarn*, *Six Dinner Sid*, and *Fry Bread,* which could engage students in wondering about mathematical concepts such as measurement, multiplication, and doubling whole numbers and fractions. Last, there are *illustration-exploring* books, which have images that provide inspiration for exploring mathematical ideas, like *I Spy Shapes in Art*, *Duck! Rabbit!*, and *The Little Blue Truck,* which shows geometric shapes, data, and counting. By keeping this framework in mind, teachers can leverage the texts they are already using as part of their reading and language arts instruction to be a springboard for math problem solving.

The National Council of Teachers of Mathematics (NCTM) has outlined effective practices for teachers of mathematics at any level (2014). When teachers use picture books to engage students in mathematical problem solving, they are using several of these effective math teaching practices. First and foremost, they are "implementing tasks that promote reasoning and problem solving" (NCTM, 2014, p.10). Importantly, literature creates those critical multiple entry points and supports students to engage in mathematical reasoning within familiar contexts. Teachers also "facilitate meaningful mathematical discourse" when using picture books in their lessons (p.10). As Shatzer (2008) noted, the contexts of picture books support students to communicate mathematically. That communication can and should take the form of both oral and written discourse as students make sense of the problem, engage in solving it, share their reasoning, and consider the reasoning of others. As these examples highlight, using picture books supports effective mathematics teaching.

Classroom Implementation

While we explored the framework for classifying texts (Hintz & Smith) above, here are some practical examples of ways to implement the use of picture books in a math lesson. You will notice that these examples come from the categories of *idea-enhancing* and *illustration-exploring* and are picture books commonly found in classrooms and/or school libraries that can promote math problem-solving. One strategy is to create a context for a problem-based lesson by using the picture book as the lesson's launch. Picture books are full of experiences that are familiar to students, including situations where their notices and wonders could be explored using math. For example, when reading *The Little Blue Truck* by Alice

Schrtle, it is hard not to wonder how many feet helped the little blue truck out of the mud! Another way to use a picture book in a lesson is to read the book in its entirety and then ask children to solve a specific problem they may be wondering about, like how many meals does Sid eat in a week in the book, *Six Dinner Sid* by Inga Moore. Teachers can also pause while reading a book to solve a problem and then finish the book seeing how the characters in the book solved the problem. For example, teachers could begin reading *Centipede's 100 Shoes* and pause to figure out how many extra shoes the centipede bought when he learned he did not have 100 feet. A fourth strategy is to let students do the problem-posing. *Extra Yarn,* by Mac Barnett, is a great example of a picture book that leads children to ask many mathematical questions. In the story, the main character, Annabelle, finds a magical box of yarn. She begins knitting sweaters and never seems to run out of yarn. When reading the book, children might wonder how much yarn Annabelle uses to make sweaters for her class or how far the box traveled to the archduke and back to Annabelle. With a little more information and a few assumptions, they can answer those questions.

Getting Started

So how do you get started? There are many collections of picture book math tasks available. Elementary math curriculum often includes suggestions for picture books embedded in lessons. Using tasks based on picture books that already exist, like the tasks in the Picture Book Library at Utah Valley University's Creative Learning Studio website, is the easiest way to get started. Most of the tasks are based on common picture books that you can find at your school or public library. Try to make sure you find a copy of the book rather than playing a video off of YouTube, as it is much more engaging for children to hear you read. Once you find the book, read it multiple times:

1. Read it to enjoy the story, then read it from your students' perspective.
2. Note where your students might get excited about the story and make some mathematical notices.
3. Read it to ensure that the problem you plan to pose is tightly aligned with the context of the book, otherwise your lesson will feel disjointed or forced.

Anticipating Student Thinking

When using the picture book as part of a problem-based lesson, consider following Smith and Stein's (2011) *Five Practices for Orchestrating Discourse*. Once you've identified the book and problem that align with your lesson's mathematical goals, it's time to anticipate student thinking. To start, solve the task you plan to pose and anticipate the strategies your students might use to solve the problem. If you are engaging in a notice and wonder lesson, you'll want to anticipate the types of problems your students might pose and what information they'll need to solve their problem.

Once you've anticipated the strategies your students might use, you are ready to prepare assessing and advancing questions for each anticipated strategy. Assessing questions help the teacher understand a student's thinking. Advancing questions help extend or deepen the students' understanding of the concept (Smith & Sherin, 2019). Since you are basing your problem from a picture book, you should purposefully connect the questions to the mathematics at hand and the story elements in the picture book to help students connect the math they are doing with the familiar context you've created. A monitoring chart (see Figure 1) is a useful tool for recording the strategies you have anticipated and the questions you plan to ask (Gearing & Disney, 2023).

Figure 1

Basic Monitoring Chart

Monitoring Chart

Strategy/Level/Solutions	*Questions to Ask*	*Students*
Cannot Get Started		

Adapted from Smith, M. & Stein, M. K. (2018). *5 Practices for orchestrating productive mathematics discussion*. National Council of Teachers of Mathematics.

Teaching the Lesson

Once you've anticipated strategies and planned purposeful questions, it's time to launch your lesson and engage your students in your picture book lesson. Before you begin to read, set the purpose for the read aloud. You might do this by giving students a particular listening job if you plan to pose a specific problem or you might ask them to write down everything they notice or wonder that they could solve using math as they listen. As you read, engage the students in both the familiar contexts of the story and the math they might notice. Use character voices, be silly, and get children excited about the story and math.

As students work on solving the problem, use your monitoring chart to record student names in the corresponding row based on the strategy they are using. Then select students to share based on a logical sequence of the strategies they are utilizing. The most concrete strategy should be shared first so that everyone can engage in the discussion part of this lesson. It's important to make sure you connect learning back to the context created by the picture book throughout the discussion (Shatzer, 2008). Once you've completed the task, you can extend learning by posing similar problems with the same familiar context. You could also ask students what they still wonder, allowing one question to lead to more worthwhile problems to solve.

Create Your Own Tasks

Once you feel comfortable implementing tasks that have already been created, start making your own! Any book can be turned into a math task, you just have to create a bank of ideas for yourself based on the picture books you frequently use in your classroom. Here are some steps to get started.

1. Start with a rich picture book that has an engaging story and purposeful illustrations. You don't have to look far because the books you are already using for reading and language arts instruction are the perfect place to start.
2. As you read, write down everything you notice and wonder that could be explored using math. Keep all of the books and all of the notices and wonders in an accessible document or notebook.
3. Make sure the tasks you pose are worthwhile and tightly aligned to the text or illustrations in the story.
4. Refer back to your list when you decide you would like to integrate a book you are already using in another content area or morning meetings into math.

No matter whether you use tasks that are already created or you develop your own, using picture books as a launch into mathematical problem solving is an effective way to engage and support your students in mathematics learning. Picture books provide students with meaningful contexts in which they can make connections to their own lives as well as the mathematics concepts they are exploring (Hintz & Smith, 2013; Shatzer, 2008). Picture books also create opportunities to explore rich mathematics tasks that require reasoning and problem solving and require students to communicate mathematically (NCTM, 2014; Shatzer, 2008). Finally, picture books are an integral part of the elementary classroom experience, so leverage the picture books you are already using as part of your reading and language arts instruction for your mathematics instruction, too!

Using *I Am One* to Inspire Mathematical Problem Solving

I have always been grateful to work with great people, especially today. One of my fellow math teachers passed along an article written about connecting picture books with math instruction, and I was able to access some math tasks from the UVU Creative Learning Studio's Math Picture Book Library, which gave me the breakthrough I needed for National Kindness Week. Based upon their recommendations, I was able to connect the message of I Am One by Susan Verde with a grade-level math standard about multiplication in a meaningful way.

Task (Standard 4.NBT.5): *Imagine how many things we could do if we worked together! If everyone in the class completed 5 (or 15, 25, 40, 55, etc.) acts of kindness, how many acts of kindness could we complete? How do you know? Use pictures or numbers to show your thinking. Then use words to explain how you know.*

Anticipating Strategies: *Students might use a base-ten model, repeated addition, a known fact, or partial products (see Figure 2).*

Figure 2

Monitoring Chart for I Am One

Monitoring Chart

Problem to Solve: *Imagine how many things we could do if we worked together! If everyone in the class completed 5 acts of kindness, how many acts of kindness could we complete? How do you know? Use pictures or numbers to show your thinking. Then use words to explain how you know.*

<table>
<tr><th>Strategy/Level/Solutions</th><th>Questions to Ask</th><th>Students</th></tr>
<tr><td>Repeated addition
34 + 34 + 34 + 34 + 34 = 170</td><td>How did you know how many times you needed to add 34?
How could you rewrite this equation to use multiplication?</td><td></td></tr>
<tr><td>Area Model<table><tr><td></td><td>30</td><td>4</td></tr><tr><td>5</td><td>150</td><td>20</td></tr></table>150 + 20 = 170</td><td>Why did you decide to break the numbers apart this way?
How do the numbers connect to the problem?
How could you use an equation to represent the area model?</td><td></td></tr>
<tr><td>Partial Products
34
X 5
20
+150
170</td><td>Where did the partial products of 20 and 150 come from?
Why can you break apart the numbers that way?
Is there a different way you could break apart the numbers?</td><td></td></tr>
<tr><td>Cannot Get Started</td><td>What is happening in the problem you are trying to solve?
What are the characters in the problem doing?
What do the characters in the problem need to find out?</td><td></td></tr>
</table>

Adapted from Smith, M. & Stein, M. K. (2018). *5 Practices for orchestrating productive mathematics discussion*. National Council of Teachers of Mathematics.

Launch: *Read I Am One by Susan Verde aloud, then ask students what they notice or wonder that could be answered using math. Then pose the task: At the end of the story, the author writes, "I can make one drop in the water…to start ripples…that become swells, then waves, traveling over oceans…across borders and boundaries…landing on distant shores to start a chain reaction, inspire a movement, make a change. One is all it takes to start something beautiful." Imagine how many things we could do if we worked* *together! If everyone in the class completed 5 acts of kindness, how many acts of kindness could we complete? How do you know? Use pictures or numbers to show your thinking. Then use words to explain how you know.*

Explore: *Students work independently or in pairs. Provide access to manipulatives as needed. As students work, use the monitoring chart to identify strategies students are using to help facilitate the upcoming discussion. Purposefully select students to share how they solved the problem based on the lesson's goal.*

Summarize: *Selected students share how they found the product - explaining their thinking. Create an anchor chart and facilitate discussion that helps students connect strategies used to reach the lesson's goal.*

Two weeks later. National Kindness Week was a smash! Students responded positively to the grade-level theme of the week, especially being able to connect the content and meaning of the book in different ways. Within the math lesson, students were excited to use larger numbers of acts of kindness - a few even challenged themselves to find out how many individual acts of kindness each student in the class would need to do to reach a million acts of kindness. Answer: In our class of 34 students plus me, each of us would need to complete 28,571 acts of kindness! We dabbled with numbers larger than we usually consider in this particular standard, but when students are into it and determined to figure it out, what can you do?

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Additional resources: UVU Creative Learning Studio - Math Picture Book Library.

A New Twist on an Old Friend: A Partitive Model for Dividing Fractions

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New Ways to Model and Represent Mathematical Thinking

The age of modeling, understanding, doing, and manipulating mathematics is here. Diagrams, pictures, manipulatives, and tools, are all different types of mathematical models (Lesh et al., 1987). Students in schools and classrooms are now, more than ever, actively constructing mathematical understanding by engaging in lessons that promote mathematical practices such as modeling, making sense of problems, or looking for and making use of structures (CCSSM, 2010), and thereby, conceptual understanding of mathematics is beginning to be more common in many mathematics classrooms around the nation. Classrooms focused on conceptual-based mathematics teaching do not target following steps, memorization, or rapid calculations. Instead, they emphasize rich experiences with mathematics that encourage problem solving and construction of number relationships. Through experiences that relate mathematics to real life, relationships, visuals, and concrete ideas, students develop a number sense that allows them to synthesize and utilize mathematics. As this type of conceptual-based pedagogy burgeons, educators look for new and innovative models that help students relate abstract concepts to concrete ideas. This article proposes a conceptual model to help students construct understanding of dividing fractions.

Dividing Fractions: A Challenging Endeavor

If you are a mathematics teacher in the 3-12 grades, you have encountered students' struggle to truly understand division of fractions. Many students, as well as pre-service teachers, fail to gain a strong conceptual understanding of division of fractions (Lamberg & Wiest, 2015). This common challenge in the mathematics education community substantiates the need for a variety of mathematical models that enhance conceptual understanding of these skills, not only for students but for teachers. As teachers ourselves, we found this struggle in our everyday teaching experiences. Students divided fractions by executing procedures but when it came to justifying their mathematical reasoning, we saw sullen faces and twittering fingers. This disparity between computation and understanding is what prompted us to investigate alternative models to represent division of fractions.

Measurement vs. Partitive Models of Division.

Lamon (2017) describes two distinct and crucial structures of division; measurement and partitive. Our students had used both structures when exploring division with whole numbers. Understanding these two structures of division with whole numbers is one key that will allow students to transfer mathematical knowledge as they develop understanding of dividing fractions. The measurement structure of division is the idea of taking equal groups out of a whole and determining how many groups were removed. This verbiage of taking groups out of a whole is the current language that is being implemented in the *Common Core Mathematics Companion* by Gojak and Miles (2016). These mathematical terms allowed our students to visualize the transfer between a physical group and an abstract thought. An example of the measurement structure of division is as follows. If you had 32 cookies and wanted to give exactly 8 cookies to as many people as possible, how many people could you give exactly 8 cookies to? You have a whole amount of 32 cookies, you take out groups of 8 cookies, and find that you can take out exactly 3 groups of 8 cookies. Many of the current fraction models used in classrooms focus on the measurement structure of division. Some of the common models for helping students conceptualize the division of fractions using the measurement structure are area models, length models, and set models.

The second division structure is the partitive model (Van De Walle et al., 2010). This structure of division depicts taking a whole amount and equally sharing it into groups, with an end goal of determining the value of **one whole group**. Consider the same cookie problem mentioned above. Instead of taking out groups of 8 cookies and giving it to as many friends as possible (measurement model), the partitive model would indicate you have 8 friends, and you have to evenly share your cookies with all of them. Partitively, you would be equally sharing the cookies between 8 people and determining how many cookies one person gets (the value of one whole group). Determining the value of each (whole) group is the critical element when we consider our conceptual partitive structure to dividing fractions.

Both structures of division are crucial to student understanding. There are numerous measurement models designed to build students' conceptual understanding. However, we struggled to find a model that directed student thinking of dividing fractions towards a partitive structure. Peer exploration and dialogue of effective lessons led us to a visual model that directed student thinking towards a conceptual partitive structure of dividing fractions.

Our Conceptual, Partitive Model for Dividing Fractions

To show you our conceptual partitive model for dividing fractions, we will start by using it to divide whole numbers and move to dividing fractions. First, let us consider the partitive structure to divide whole numbers. Take a simple expression such as 24÷3. In the partitive mindset, we would ask ourselves "If I evenly share 24 into 3 groups, what is the value of one whole group?" (Figure 1). In this case you will see that 1 group will have a value of 8 after all have been passed out.

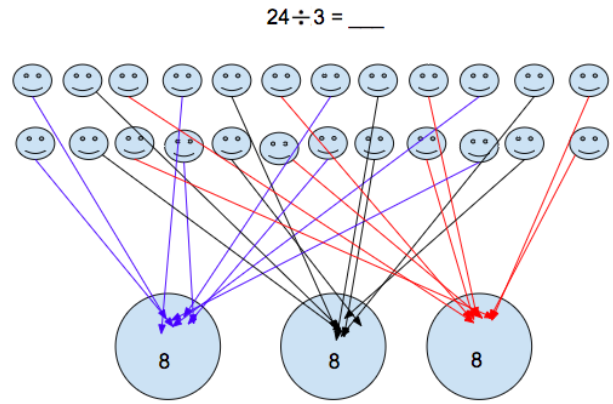

Figure 1: Evenly sharing 24 into 3 groups.

Now, simply, using the same definition of partitive division, let us take this model one step further to help us divide a whole number by a fraction. Consider the problem, $5\div\frac{1}{2}$. I would always encourage the students to repeat the meaning of partitive division by saying "If I share 5 into $\frac{1}{2}$ of a group, what is the value of one whole group?" (Figure 2) When our students passed out all 5 into $\frac{1}{2}$ group they initially got stuck. There was not one whole group in which to find the value, they only had $\frac{1}{2}$ group. By definition of this partitive structure of

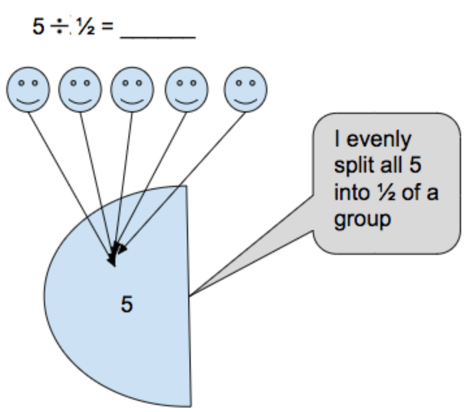

Figure 2: Evenly sharing 5 into ½ group.

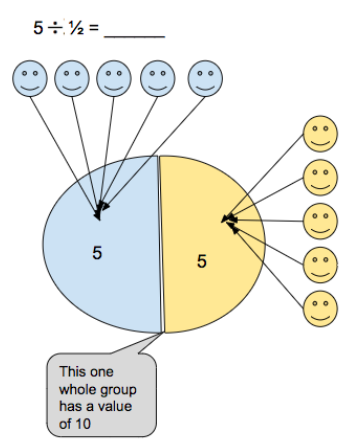

Figure 3: Completing one whole group, by recreating another ½ group.

division however, they needed to find the value of one whole group. At this point, we introduced a key question to guide student thinking. “How much more of the group do you need to complete the whole group?” After investigation, the students discovered that if they added another $\frac{1}{2}$ group, they were able to complete the whole group. As they reflected on the whole group, they noticed it was made up of two ½ groups, each having a value of 5 (Figure 3). Students concluded the value of the one whole group being 10, so $5 \div \frac{1}{2} = 10$

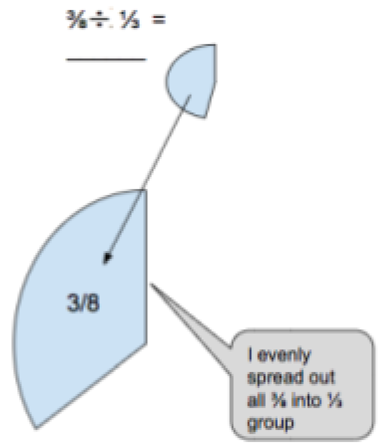

Figure 4: Evenly sharing 3/8 into 1/3 group.

Finally, students are introduced to a division problem that poses a fraction being divided by a fraction. Let us consider the problem $\frac{3}{8} \div \frac{1}{3}$. The critical component is going all the way back to our fundamental understanding of the partitive structure of division. Again, students repeated the meaning of the partitive structure by saying “If I share $\frac{3}{8}$ into $\frac{1}{3}$ of a group, what is the value of one whole group?” Using this conceptual partitive model allows an individual to see exactly where the values are being shared. In Figure 4, the student has no choice but to share all $\frac{3}{8}$ into the $\frac{1}{3}$ of a group. The students noticed that again there was not a whole group. The key question was posed to the students “How much more of the group do you need to complete one whole group?” Soon, students decided to supplement the existing $\frac{1}{3}$ group with 2 more $\frac{1}{3}$ groups. Now, considering that the initial $\frac{1}{3}$ group had a value of $\frac{3}{8}$, proportionality requires that each subsequent $\frac{1}{3}$ group would have a value of $\frac{3}{8}$. By adding 2 more $\frac{1}{3}$ groups, each with a value of $\frac{3}{8}$, students created one whole group and found the value as such, $\frac{3}{8}+\frac{3}{8}+\frac{3}{8} = \frac{9}{8}$ or 3 groups of $\frac{3}{8}$.

We used a sample problem with a group of 5th grade students in order to document student thinking as well give us insight into their understanding of this conceptual model. The sample problem read "Mrs. Ross wanted to put up fence around her whole round pasture. She had $\frac{3}{8}$ of a mile of fencing. Unfortunately, the $\frac{3}{8}$ of a mile of fence only completed $\frac{1}{3}$ of the perimeter. How much fence does Mrs. Ross need to complete the distance around the whole pasture? As one student was working through the problem, she drew a circle and segmented the perimeter off into three equal portions and said "So you need two more, because if it's a third, there are three parts. This is one part, this is one $\frac{3}{8}$, and this is another $\frac{3}{8}$." Each time she indicated the $\frac{3}{8}$ portion, she used her finger to point to the corresponding $\frac{1}{3}$ of the pasture. When the student was asked how she knew there would need to be 3 of the $\frac{3}{8}$, she responded "Because I just think of thirds as always being three pieces so since she had $\frac{3}{8}$ of a mile and that was $\frac{1}{3}$, I knew there had to be another $\frac{3}{8}$ and another $\frac{3}{8}$ because ummm, because otherwise they wouldn't be thirds." This student demonstrates this type of partitive thinking because she is working to complete the whole group by reconstructing portions of the unfinished group and each portion must have the same value.

Unit Fractions vs. More Challenging Fractions

You may have noticed that the fraction problems above all have a unit fraction divisor, which make it seem simple. You also may be wondering if this model still works for more challenging problems such as $\frac{6}{8} \div \frac{2}{7}$? The simple answer is a resounding yes! This article will not spend too much time explicating every problem type, that is for you to play with, but when using this conceptual partitive model for a problem like $\frac{6}{8} \div \frac{2}{7}$ students can employ their proportional reasoning to solve them. Students would start by sharing all the $\frac{6}{8}$ into $\frac{2}{7}$ group. They would try to complete the entire $\frac{7}{7}$ group (one whole group), and thereby would need to add on $\frac{5}{7}$ group. Considering $\frac{2}{7}$ group had a value of $\frac{6}{8}$, then proportionality would require $\frac{1}{7}$ group have a value of $\frac{3}{8}$. Therefore, students would see that the additional $\frac{5}{7}$ would have a value of 5 x $\frac{3}{8}$, with a total value of one whole group as: the starting $\frac{6}{8}$ + the additional (5 x $\frac{3}{8}$) = $\frac{21}{8}$. If you are just thinking, 7 x $\frac{3}{8}$, you are right on! Think how interesting of a way that is to solve the problem $\frac{6}{8} \div \frac{2}{7}$! It does not exactly look a lot like the old $\frac{6}{8} \times \frac{7}{2}$ method, now does it!

Instructional Suggestion – Paper Plate Partitioning

We have found one way to introduce this conceptual partitive model concretely, which is by using paper plates. We have all used the classic paper plate technique for equal sharing division problems, so why not extend them to equal sharing division of fraction problems! As you can already imagine, the only resources you need to get started are pencils, paper plates, scissors, and fraction bars/bits of paper to write on. Simply start with a progression of problems similar to those outlined above. Start with division of whole numbers and have students solidify the understanding that the answer to this kind of conception of division is the value of one whole group – or the value of one whole plate. Soon, you will have students sharing amounts into ¼ of a paper plate, and then using proportionality to try and figure out what the value of one whole paper plate is. For example, if you posed the problem $\frac{2}{3} \div \frac{1}{4}$ the students would pass out all the $\frac{2}{3}$ to the only $\frac{1}{4}$ of a plate (Figure 5). Then, using proportionality and guided by the question "what is the value of one whole plate?" they complete the whole plate by adding three more $\frac{1}{4}$ plates, each with a value of $\frac{2}{3}$ (Figure 6). Thus, the value of one whole plate is $\frac{8}{3}$

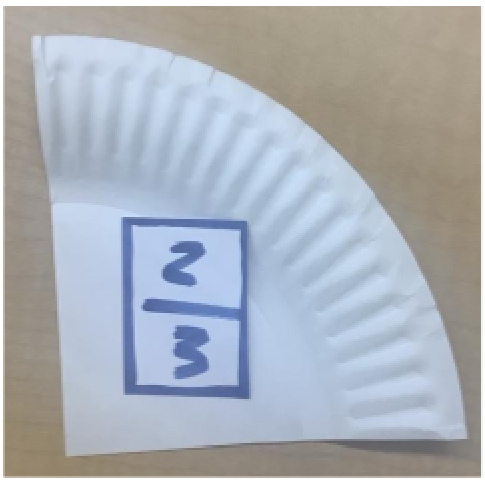

Figure 5: Evenly shared onto the only ¼ plate.

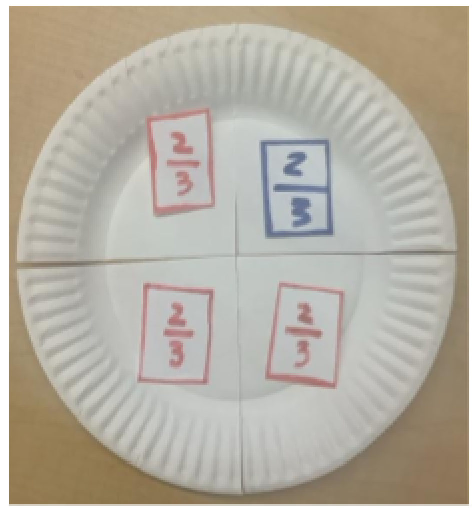

Figure 6: Recreate the whole plate, adding 3 more rds.

Relevance

This partitive structure represents the traditional algorithm more closely than any model we have yet encountered. When teachers instruct to, "copy, dot, flip" or, "turn it upside down and multiply," you will find that the procedure closely follows the mathematical operation and reasoning that is demonstrated by this partitive model. Consider the aforementioned problem $\frac{3}{8} \div \frac{1}{3}$. When you arrive with one whole group and you are ready to determine the value, you are ultimately taking $\frac{3}{8}$, 3 times. If you were to flip $\frac{1}{3}$ upside down (take the reciprocal), and multiply it by $\frac{3}{8}$, you have arrived at the same calculation that the conceptual partitive model delineated, $\frac{3}{8} \times 3$.

This conceptual partitive structure for dividing fractions has many implications on instruction in the 3-12 classroom. It is not uncommon for students to struggle with conceptual understanding of reasoning with rational numbers (Smith III, 1995). In fact, it is one of the more challenging operations and concepts in K-12 education (Siegler & Lortie-Forgues, 2017). This model may prove valuable as the mathematics education community continues to pursue new ways to link pictorial understanding to abstract concepts. Furthermore, we see power in this model when we look at students' work and listen to their justifications. Students are developing links between this model of division and conceptual understanding of fractional division. These links from pictorial representations to abstract, and from prior knowledge to new content, are links that are essential for learning.

So, for all you mathematics teachers who are intrigued, and saying "does that really work with all fractions?" or "how have I not thought of using the partitive model in this way before?" Go ahead, give it a try, draw a picture, use different whole numbers and fractions, explore and have fun. See what your students think about this model and please email us to let us know how it went and what you think!

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An Alternative Chip Model for Operations on Integers

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Abstract: The use of positive and negative colored counters has been used to represent the four basic operations as a visualization tool in elementary education. However, the previous models are inefficient and inconsistent with various algebraic definitions and also leave little room for modeling division. We seek to remedy this situation.

Keywords: operations on integers, integer chip model, K-6 mathematics education

Operations with integers are a fundamental piece of K-6 mathematics education. Teaching the four main operations can already be a fairly tricky concept, which only gets exacerbated when using the set of integers. Recall that many teachers use manipulatives to lead elementary students from concrete to the abstract understanding of integers (Stein & Bovalino, 2001; Cope, 2015). One model commonly used is the integer chip model, which textbooks and other teaching resources endorse (Long, DeTemple, & Millman, 2015; Fierro, 2013). We seek an efficient method to use colored counters to represent all four basic operations consistently while maintaining alignment with their fundamental definitions.

Initially, we noticed many texts and math manipulatives are limited in their ability to accurately portray anything beyond addition and subtraction. Some representations are sufficient regarding multiplication, but many authors tack on division as an afterthought, relying on simple knowledge recollection of integer multiplication fact families to construct a seemingly unrelated and weak division model. There have been limited attempts at creating a solid division chip model, but they are scarce and have not gained traction (Lamb, Bishop, Philipp, Whitacre, Schappelle, 2018; Battista, 1983). We seek to update and humbly upgrade this model for consistency, aligning with the foundation of algebraic definitions rather than elementary education tropes. In short, we not only believe our design is simpler, but it is also more consistent with the building blocks of modern-day algebra.

To review, the chip model consists of colored counters, simple white and black circles, representing +1 and -1, respectively. In Figure 1, we show how to represent 0, and then 3 in two different ways, for those readers not familiar with the concept of integer chips. Essentially, these numerous ways of displaying the same value can be attributed to the additive identity. We want to use prior definitions of the existence of negatives: that is, $m + (-m) = 0$. We will define *zero pairs* as the pairing of one white chip and one black chip to represent 0.

Figure 1

Three chip representations for 0, 3, and 3, respectively

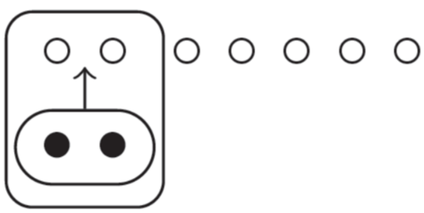
These objects give students something tangible to manipulate as they attempt to grasp the idea of negative numbers before performing operations on them. A combination of details such as where the chips are placed, the chip color, and the movement of the chips, are usually discussed.

Addition and Subtraction

We will review the common process of adding and subtracting under this model. For addition, we push two separate groups of chips together (the addends), and for subtraction we remove the appropriate amount of chips (the subtrahend) from the starting amount (the minuend). In Figure 2, we represent the chip model of 7 + (-2). Notice that each pair of white and black chips "zeroes out" in the result, leaving a sum of 5.

Figure 2

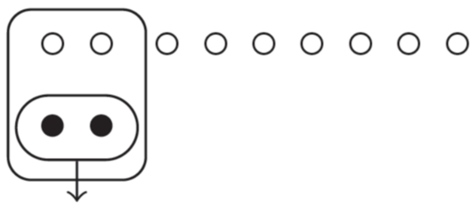
Representation of 7 + (-2)



In Figure 3, we represent the chip model method for 7 – (-2). Notice that if we were to represent 7 with all white chips, there would be no black chips available in order to remove two of them. Because of this, we use the existence of negatives and include two pairs of 0 so that we have enough blacks to remove. Now, we have enough to physically take away two black chips. We can now see the difference is 9. The diagram is similar for subtraction compared to the first addition problem in Figure 2 because these are inverse operations.

Figure 3

Representation of 7 – (-2)



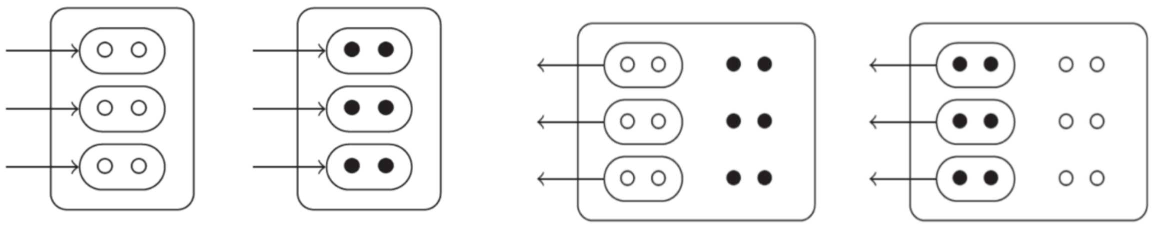
We do not take issue with these similar forms of representing these two basic operations the way we have seen in many teaching resources, as the act of combining two groups is fundamental to addition while removing some objects from a set of objects aligns with subtraction. However, as we will show in the next section, we think that the common representation of multiplication (and the general lack of for division) is not a natural way of thinking in the same way.

Original Multiplication

In order to compare our proposed method of representing multiplication and division with chips, we should first review the typical textbook methods. In Figure 4, notice the direction arrows shown for these models. An arrow going into the box represents the positive motion of inclusion, while an arrow going out of the box represents the negative motion of removal. While we appreciate this consistency, it heavily relies on the elementary education notion that multiplication is only representative of "skip counting" addition.

Figure 4

Chip representations for $3 \cdot 2$, $3 \cdot (-\ 2)$, $(-\ 3) \cdot 2$, *and* $(-\ 3) \cdot (-\ 2)$, *respectively*



For example, in the fourth image in Figure 4 we remove three sets of chips and there are two black chips per set, thus showing the operation $(-\ 3) \cdot (-\ 2)$. In all of the above examples, we are required to start with 0 inside the box, and then put chips inside or take chips away. There are many problems with this. First, while it works for the subtraction model to include enough extra chips for there to be enough to remove and carry out the operation, in multiplication this can require so much extra work in setting up the problem before even carrying out the procedure, requiring an excess amount of chips and confusing the student.

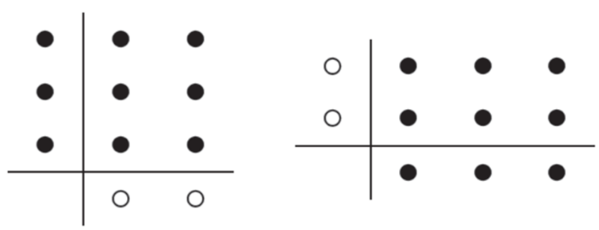
Second, the multiplicative identity is 1, not 0, and so building a multiplication problem from a 0 starting point does not make mathematical sense if you are thinking of multiplication as a standalone operation.

Third, this model requires a separate case drawing for all four sign configurations; quite a bit for young learners to remember. Fourth, if a user tries to invert the process to create a division representation, using 0 as the initial condition presents plenty of issues. This is probably why many division models look entirely different in most texts, relying on prior knowledge of fact families to build a multi-colored array, such as in Figure 5. This will be different than ours in many ways, particularly given this model does not continue with the inclusion/removal motions that were a large aspect of the first three operations. In doing so, this model then appears unrelated, and instead relies on memorization of multiplication facts. While the Figure 5 model rehashes the multiplication model, ours instead will require a process that discovers the resulting quotient, and naturally incurs the white/black color of the result.

Fifth, in the prior subtraction model, the minuend (first value in the expression) represents how many chips to deal with initially, whereas in the multiplication model, the multiplier (first value in the expression) represents the motion of removing or including certain-sized groups of chips; we find this inconsistent as well.

Figure 5

Typical integer division model

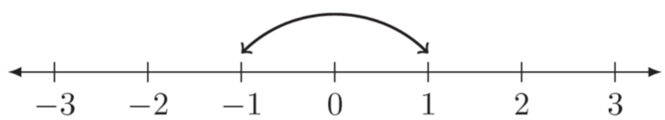


Proposed Multiplication Model

We introduce a new multiplication model that evokes less confusion about arrow direction, entirely avoids the need to supply enough chips to "start" from zero, and considers fewer cases. Our model will be invertible and also provide a natural chip model for division. We revisit Figure 4, but more efficiently. First, we define what we call a *flip*. By the geometric definition of a reflection, this is a transformation that preserves distances. Algebraically, a flip represents the multiplication/division of a value by -1, which becomes its additive inverse—flipping chips tracks this process. Using a traditional number line, such as in Figure 6, we can see that a reflection, in this case across the origin, simply interchanges positive and negative rays that make up the number line. This can be extended to a group of chips, which is represented by a single integer, because this is a natural pairing between a positive integer and its negative counterpart. In the past, we have typically described this concept as "sets of" or "groups of" in multiplication algorithms.

Figure 6

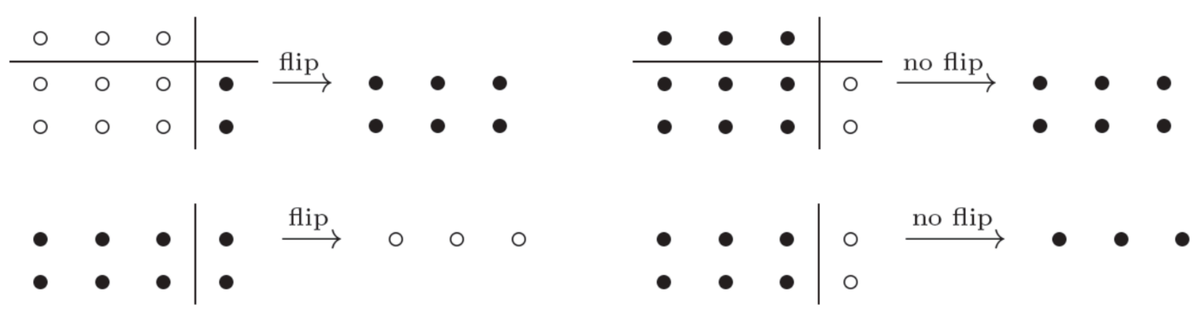
Illustration of reflections across a number line, and existence of negatives



Keeping this in mind, we present our multiplication and division models for integers in Figure 7. Contrasting this with Figure 4, there are many differences. We will elaborate upon the process shortly.

Figure 7

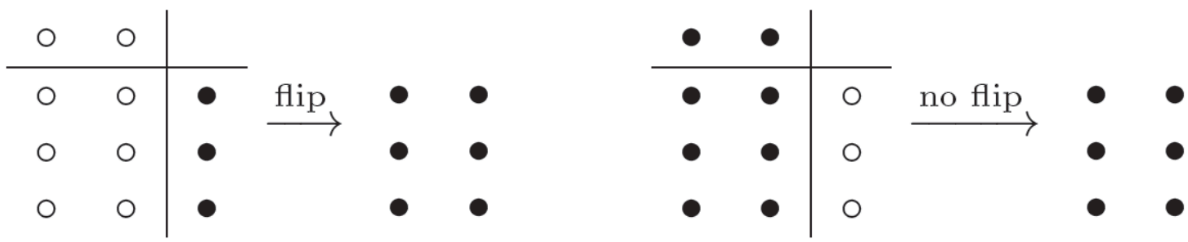
Top left: $3 \cdot (-\ 2)$*, top right:* $(-\ 3) \cdot 2$*, bottom left:* $(-\ 6) \div (-\ 2)$*, bottom right:* $(-\ 6) \div 2$



We will now elaborate upon the procedure for our multiplication model first. Suppose that we wish to demonstrate the product of a negative and a positive, where *A* and *B* are integers $(A, B \in Z)$. We begin by arranging *A* chips horizontally (recall that a white chip represents +1 and a black chip represents -1). Adjacent to the horizontal chips, we arrange *B* chips vertically. We now build an array (Figure 8). For every *B* chip, we arrange a set of *A* chips horizontally, extrapolating the same color.

Figure 8

Representations for $A \cdot (-\ B)$ *and* $(-\ A) \cdot B$*, respectively*



Please note that the arrows do not have any directional significance other than pointing to the final product (and later, final quotient). We then count the resulting *B* rows of *A* chips. This will result in a number *C*, the product, which represents $A \cdot B$, up until the prospective sign. Lastly, we determine this sign. If the *B* chips are white, then $A \cdot B\ =\ C$. If the *B* chips are black, then we physically flip the entire array of chips and $A \cdot B\ =-\ C$. Conceptual knowledge here relies on students already understanding the definition of a flip and how it is illustrated in Figure 6. This procedure uses less chips and decision making than the classical chips multiplication model. Furthermore, it naturally showcases multiplication by zero (Figure 9) and algebraic properties of multiplication such as commutativity (Figure 10). It makes sense to have a distinguished model for multiplication as it is an entirely separate operation on its own, independent of addition. The classic model tries to tie the multiplication model directly to the addition model through skip counting.

Figure 9

Chip representation for $3 \cdot 0$

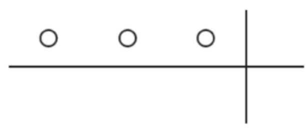
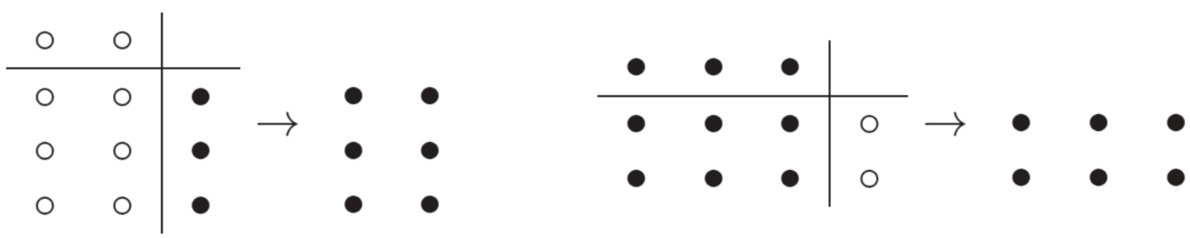


Figure 10

Showcasing commutativity of the multiplication model



A salient feature of this proposed multiplication model is the fact that it is invertible. Thus, it provides a more intuitive model for division that is otherwise absent from the literature. Suppose we wish to demonstrate the operation $A \div B$ (Figure 11). We begin by arranging *B* chips vertically (again, using the appropriate color depending on the sign of *B*). Then, we partition *A* chips into groups of equal size to the left of *B* chips so that each *B* chip corresponds to a single group. Let *C* denote the number of chips in such a group. If *B* chips are white, then $A \div B = C$. If *B* chips are black, then we must flip the chips in the group and the result is $A \div B = -C$.

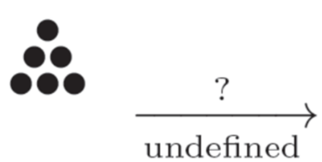
Figure 11

Representations for $(-\ 6)\div(-\ 3)$ *and* $(-\ 6)\div3$

A notable feature of the division model is demonstrating the abstract idea that division by zero is undefined. This is essential, and often not explained in elementary classrooms either with manipulatives or actual values; instead just accepted as fact. If $B\ =\ 0$, it does not make sense to attempt partitioning *A* chips above into 0 groups (see Figure 12). However, dividing 0 by anything does work because partitioning 0 chips into as many groups as necessary is doable, even if tedious.

Figure 12

Inability to partition A chips to 0 groups



Supporting our New Model

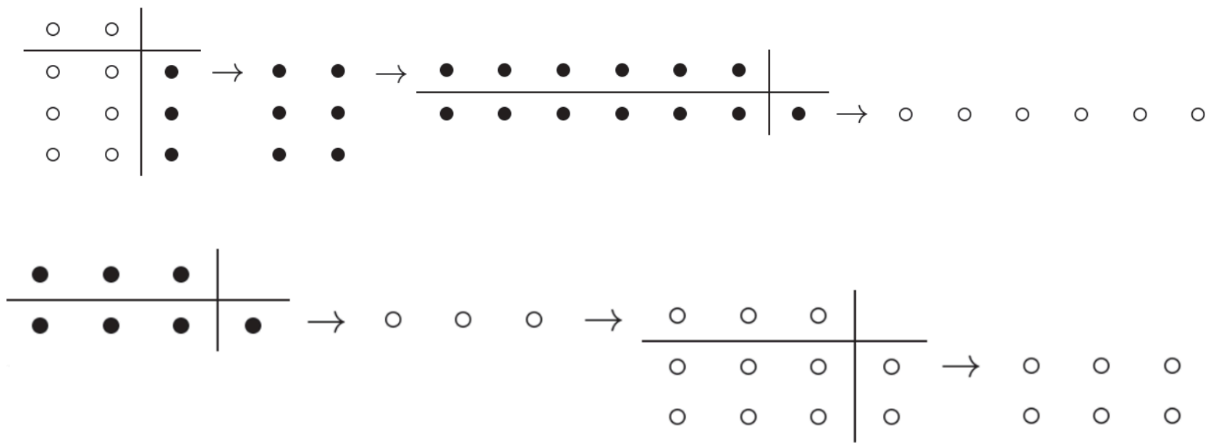
Notice that in our model, the main focus is to flip or not to flip. We believe the process of flipping is more definition-based and intuitive than the binary decision (arrows in, arrows out) in the traditional multiplication model. The traditional model also requires the extra step of deciding on the inclusion of more chips before even proceeding with the problem in the first place. We can imagine the user would find it frustrating to first set up the entire problem before actually carrying out the procedures. Recall, in the previous model the user also has to consider four separate cases, such as $-\ A\cdot B$, $A\cdot\ -\ B$, $A\cdot B$, and $-\ A\cdot\ -\ B$ (see Figure 4). In our newer proposed model, the decisions are much fewer and straightforward (see Figure 7).

Additionally, our division can be done similarly, inverting the process. Previous models rely on various combinations of the products of the units, and then extrapolating into appropriate arrays that are part of the overall multiplication fact family (again, refer to Figure 5). To us, this was not a true division model that made sense as a standalone. To be mathematically sound according to ring theory, addition and multiplication should have their own unique models that can have the process inverted to create natural subtraction and division models. To be clear, we are essentially proposing two separate models, one for adding/subtracting integers, and one for multiplying/dividing integers. However, these models are much more related.

Our model, in a sense, relies on arrays the entire way, regardless of operation. Because of this, Figure 10 shows commutativity is very clear. We extend this idea and provide a demonstration for associativity as follows in Figure 13. We start by showing $2\cdot(-\ 3)$, and then multiply by (-1); we see that $[2\cdot(-\ 3)]\cdot(-\ 1)\ =\ 6$. Next, we shift the parenthesis, showing associativity. So, $2\cdot[(-\ 3)\bullet(-\ 1)]$ is represented by $(-\ 3)\cdot(-\ 1)$, and then multiplying by 2. Compare how algorithmically our process works efficiently in multiple steps rather than pausing and reorganizing to include and remove the appropriate chips needed to proceed, which would unfortunately be necessary to succeed with the traditional model. We think this is one of the biggest advantages in our model, and becomes more streamlined as a result.

Figure 13

Showcasing associativity of the multiplication model, but output of -6



Summary

Manipulatives are a necessary tool for bringing students from the concrete to the abstract in learning the fundamentals of mathematics at a young age. It is imperative to have models that are both simple and effective. we provide an update to previous model attempts, revamping the process regarding efficiency, consistency, and accuracy. We outlined many strengths to our model compared to any predecessors. While we had no complaints about the addition and subtraction integer chip models, we found many faults with multiplication, which we outlined above. The division model is practically non-existent in the literature so we also extended our multiplication model to fill this void. While there are still two separate overall models, they are more compatible and connect with each other. We have several examples and illustrations for both students and teachers to follow, and invite teachers to try these models out with their students, and report back any findings after further exploration, to determine to what extent this model can help students truly conceptualize the relationships between integers.

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Discourse About Student Discourse

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Lindsey Henderson, USBE

Picture the following: a math class where all of the students are silently working through pages of calculations and equations. A room devoid of discourse and debate. A room where students independently struggle to grasp mathematical concepts.

Now what if, instead, we create classrooms where mathematical language and interactions are predominant throughout the room? This discourse or "mathing elixir" becomes the catalyst for more engaging and successful mathematical learning moving students away from a learning desert to a richly forested place of progress.

According to the National Council of Teachers of Mathematics, the facilitation of meaningful mathematical discourse is one of eight evidence-based teaching practices: "Effective teaching of mathematics facilitates discourse among students to bring shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments" (2014). A good teacher recognizes and uses discourse to lead students on a journey of learning, asking questions and engaging students in dialogue that sparks creativity and encourages students to think independently. "The teacher's skilful questioning…helps students to identify thinking processes, to see the connections between ideas and to build new understanding as they work their way to a solution that makes sense to them" (*Asking Effective Questions*, 2011).

I like to think of discourse as a "twofer." "When a teacher engages students in meaningful mathematical discourse, they are also engaging in several other effective teaching practices–posing purposeful questions, eliciting and using evidence of student thinking, using and connecting mathematical representations, and supporting productive struggle" (Smith, Stein, 2011, p. 1). These are some of the reasons Utah's secondary mathematics community has embraced student discourse as a high priority, referencing its power many times in Utah's Core Standards for Mathematics, Utah's Secondary Mathematics Vision as well as Utah's PK-12 Mathematics Framework.

So what does the discourse "mathing elixir" look like in Utah mathematics classrooms? A good teacher will create a learning environment in which student discourse is a normal and expected element of learning. Discourse should be encouraged and respected, and students should be given opportunities to share their ideas and reason with peers as well as the teacher. If a student has a great idea or creative solution, they should be given the chance to explain it (even if the answer is incorrect) and the class should be encouraged to respectfully comment and to make connections to what was shared. Utah's Core Mathematics Standards, in every grade from kindergarten through secondary mathematics III, explicitly calls out the power of students engaging in "constructing viable arguments and critiquing the reasoning of others" (K-SM3.MP.4).

Utah's PK-12 Mathematics Vision and Utah's PK-12 Mathematics Framework both emphasize the importance of discourse and provide resources to help educators implement discourse routines in classrooms. Three powerful USBE sponsored book studies, *Making Number Talks Matter, Building Thinking Classrooms, and 5 Practices for Orchestrating Productive Mathematics Discourse*, highlight routines that can help educators begin to support discourse in their mathematics classrooms.

Discourse can also be used as a powerful way to assess, differentiate, and offer just in time supports in a student's mathematical journey, helping a skilled educator to identify areas of understanding or challenge and with little to no papers for a teacher to grade. It is also a great way for students to learn from their peers and build relationships and a wonderful way for teachers to give immediate and meaningful feedback to student's mathematical ways of thinking and knowing. According to Eric Jensen "newly processed information should be given immediate feedback (every ten minutes) for correction to occur before the information gets too fixed" (2005).

So next time you're in a Utah mathematics classroom, remember to look for, encourage, and use the discourse mathing elixir. It is one of the key ingredients to a successful mathematics learning experience where students can tap into the joy of mathematics as an active participant in the learning experience. In the article *Never Say Anything a Kid Can Say*, Steven Reinhart states: "...by merely telling them [students] answers, doing things for them, or showing them shortcuts, I relieve students of their responsibilities and cheat them of the opportunity to make sense of the mathematics they are learning, I must ask good questions, allow students to struggle…" (2000). We challenge each of you this school year to allow each of your students the opportunity to feel the joy of mathematical connections and discovery through rich student discourse!

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How a Good Grade can Affect our Math Students

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Abstract: The forthcoming article is rather an occasion to provoke a discussion among Utah math educators about how willing they really are to assign grades not based solely on ability.

Introduction

Teaching math is our job and ways to improve it pose an everlasting target. Among our duties lies undoubtedly the grading part. How many times have we encountered students who are opposed to and afraid of what they perceive a bad (low) grade to be.

By adopting what I call a *persistence-based grading* approach, I came to understand how pivotal in the whole teaching and learning process a good grade proves to be. When I first adopted this approach I was skeptical enough, but I realized assigning good grades became a catalyst for attaining a new level of communication with the students.

When generating the idea for persistence-based grading, I remained loyal to the sociocultural theory of Vygotsky (Vygotsky, 1978). The composite word sociocultural explains by itself the gist of the theory. Any learning outcome is a result of a social process, produced in and for the society. We, math teachers, have a miniature form of this society before us: the classroom.

A fundamental component of this sociocultural environment is language. As Vygotsky himself articulates it “language, the very means by which reflection and elaboration of experience takes place, is a highly personal and at the same time a profoundly social human process” (Vygotsky, 1978, p.126). Language is how we communicate ideas between teachers and students, and the meaning of the language comes from our interactions within the society we live (e.g., parents, friends, family, etc.).

Mathematics is considered to be a hard subject, and perceived as loathsome for a great many students. As a result, a plethora of students have low or no expectations for achieving some kind of success in math. I then wondered whether a good grade might be enough to fuel their interest and bolster some kind of enthusiasm for mathematics. Perhaps a good grade would be like offering joy to disheartened and disappointed people, by providing them with evidence that their efforts in a math course can bear fruit in the form of a good grade. Instead of prioritizing whether they considered all restrictions in an equation or the numerical mistakes they committed trying to solve a problem. For example, when missing some restrictions, assigning to them a satisfactory portion of the whole grade. Making a step upwards from a low level should and can be rewarded in some way, and in my opinion, this materializes with a higher grade than the usual norms for grading would have allowed.

Does this form of teachers’ demeanor pose an occasion of injustice for talented and gifted students? I need to mention here that the lenient grading policy applies to all students. Gifted students will continue to score high and have their own set of exercises to be occupied with. Simply, as the society outside the classroom has space for everyone, regardless of her aptitudes and talents, by the same token classroom society should have room for students who are ill-disposed towards math. A feasible way to achieve that is what I called before a persistent-based grading approach, wherein the good grade assigned to them carries weight. My experience has demonstrated that straight A students come to accept this kind of teachers' behavior and at the same time demonstrate a solidarity towards the students, who need that kind of reward. Solidarity is a fundamental concept for the society in general, much less in a classroom's realm.

I will be really thrilled if my above sort of commentary can precipitate a dialogue among Utah math educators. Especially if it raises awareness on what I tried to introduce as the persistence-based grading approach, based on an effective sociocultural tool, the grade, the reward in our school environment.

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